Correlation Functions and Vertex Operators of Liouville Theory

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Abstract

We calculate correlation functions for vertex operators with negative integer exponentials of a periodic Liouville field, and derive the general case by continuing them as distributions. The path-integral based conjectures of Dorn and Otto prove to be conditionally valid only. We formulate integral representations for the generic vertex operators and indicate structures which are related to the Liouville S-matrix.

Keywords: Two-dimensinal conformal field theory; Liouville theory; Canonical quantisation; Vertex operators; Correlation functions

The Liouville theory has fundamentally contributed to the development of both mathematics [1, 2] and physics [3], and beyond it fascinated with a wide range of applications as a conformal field theory. Nevertheless, its quantum description is still incomplete. By canonical quantisation [4]-[8] it could be shown that the operator Liouville equation and the Poisson structure of the theory, including the causal non-equal time properties, are consistent with conformal invariance and locality [6, 8], but exact results for Liouville correlation functions remained rare [9] despite of ambitious programmes [10, 11].

In this letter we calculate correlation functions for vertex operators with generic exponentials of a periodic Liouville field. The vertex operators are given in terms of the asymptotic *in*-field of the Liouville theory [12], and we formulate for them an integral representation as an alternative to the formal but still useful infinite sum of [6]. However, there is so far no reliable recipe to use such integral operators directly since the complex powers of the screening charge operators describing them are not constructed yet, a problem related to the exact knowledge of the Liouville S-matrix. We calculate therefore first correlation functions for vertex operators of [6] with negative integer exponentials and continue the result analytically as a distribution, as is required by the zero mode contributions of the Liouville theory [13]. We prove so that the correlation functions suggested in [14] are conditionally applicable only. This is indeed a surprise because the conjecture of [14] was obtained by standard analytical continuation of a path-integral result for minimal models [15], which describes nothing but a special part of the operator based correlation function [9]. In this respect it is worth mentioning that already the Liouville reflection amplitudes [16] proved to be identical with those obtained from the Liouville S-matrix [13, 17].

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We parametrise the vertex operators by a free-field which allows to avoid the use of quantum group representations, and we define the approach on which both, the derivation of the correlation functions and the formulation of the integral representation of the vertex operators are based. The structures needed to understand the Liouville S-matrix are indicated, and in the conclusions we stress the importance of the results for the related WZNW cosets.

1 Free-field parametrisation

We use minkowskian light-cone coordinates $x = \tau + \sigma$, $\bar{x} = \tau - \sigma$ and select from ref. [1] that general solution of the Liouville equation

$$\varphi = \frac{1}{2} \log \frac{A'(x)\bar{A}'(\bar{x})}{[1 + \mu^2 A(x)\bar{A}(\bar{x})]^2}, \tag{1}$$

which has a particularly utilisable physical interpretation. For periodic boundaries

$$\varphi(\tau, \sigma + 2\pi) = \varphi(\tau, \sigma), \tag{2}$$

one can parametrise the non-canonical and quasi-periodic parameter functions $A(x)$, $\bar{A}(\bar{x})$

$$A(x + 2\pi) = e^{\gamma p} A(x), \qquad \bar{A}(\bar{x} - 2\pi) = e^{-\gamma p} \bar{A}(\bar{x}), \tag{3}$$

by the canonical free field $2\phi(\tau, \sigma) = \log\, A'(x)\bar{A}'(\bar{x})$ with standard mode expansion

$$\phi(\tau, \sigma) = \gamma q + \frac{\gamma p}{2\pi}\tau + \frac{i\gamma}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{a_n}{n}\, e^{-inx} + \frac{i\gamma}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{\bar{a}_n}{n}\, e^{-in\bar{x}}, \tag{4}$$

and the chiral decomposition

$$\phi(\tau, \sigma) = \left(\frac{1}{2}\gamma q + \frac{\gamma p}{4\pi} x + \phi(x) \right) + \left(\frac{1}{2}\gamma q + \frac{\gamma p}{4\pi} \bar{x} + \bar{\phi}(\bar{x}) \right). \tag{5}$$

Note the rescalings of the fields ϕ and φ by the Liouville coupling γ. We obtain so the canonical transformation between the Liouville and the free-field of [6] (γ there is 2γ here!)

$$e^{-\varphi(\tau,\sigma)} = e^{-\phi(\tau,\sigma)} + \mu^2 e^{-\phi(\tau,\sigma)} A(x)\bar{A}(\bar{x}), \tag{6}$$

and by integrating $A'(x)$ (correspondingly $\bar{A}'(\bar{x})$) using (3)

$$A(x) = \frac{e^{\gamma q}}{2 \sinh \frac{\gamma p}{2}} \int_0^{2\pi} dy\, e^{\frac{\gamma p}{2} \left(\epsilon(x-y) + \frac{y}{\pi} \right) + 2\phi(y)}. \tag{7}$$

$\epsilon(z)$ is the stair-step function, and as preconceived, the non-vanishing parameter p of the hyperbolic monodromy relation (3) becomes identical with the momentum zero mode of the free field (4), and we choose $p > 0$ [4].

Since for asymptotic 'time' τ the two terms of the Liouville exponential (6) behave as

$$e^{-\phi(\tau,\sigma)} \sim e^{-\frac{\gamma p}{2\pi}\tau}, \qquad \mu^2 e^{-\phi(\tau,\sigma)} A(x)\bar{A}(\bar{x}) \sim e^{\frac{\gamma p}{2\pi}\tau}, \tag{8}$$

they can be interpreted as *in*-coming respectively *out*-going contributions so that

$$e^{-\varphi(\tau,\sigma)} = e^{-\phi_{in}(\tau,\sigma)} + e^{-\phi_{out}(\tau,\sigma)}. \tag{9}$$

As a consequence the chosen canonical free field (4) has a physical meaning, it is the asymptotic *in*-field of the Liouville theory [12], and the *out*-field which is given by the *in*-field too

$$e^{-\phi_{out}(\tau,\sigma)} = \mu^2 e^{-\phi(\tau,\sigma)}\, A(x)\, \bar{A}(\bar{x}), \tag{10}$$

defines likewise the classical form of the Liouville S-matrix.

Two related forms of the Liouville exponential are relevant for canonical quantisation, the formal expansion in powers of the conformal weight zero functions $A(x)\bar{A}(\bar{x})$ [6]

$$e^{2\lambda\varphi(\tau,\sigma)} = e^{2\lambda\phi(\tau,\sigma)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(2\lambda+m)}{\Gamma(2\lambda)} [\mu^2 A(x)\bar{A}(\bar{x})]^m, \tag{11}$$

and the integral representation for positive λ

$$e^{2\lambda\varphi(\tau,\sigma)} = e^{2\lambda\phi(\tau,\sigma)} \int_{-\infty}^{+\infty} dk\, \frac{\Gamma(\lambda+ik)\,\Gamma(\lambda-ik)}{2\pi\Gamma(2\lambda)} \, [\mu^2 A(x)\bar{A}(\bar{x})]^{-(\lambda+ik)}. \tag{12}$$

The last equation follows from the Liouville solution (6) by using the Fourier transformation of $(2\cosh y)^{-2\lambda}$ [18]

$$\int_{-\infty}^{+\infty} dk\, \frac{\Gamma(\lambda+ik)\,\Gamma(\lambda-ik)}{2\pi\Gamma(2\lambda)}\, e^{2iky} = \left(e^{-y}+e^{y}\right)^{-2\lambda} \qquad \text{for} \qquad \mathrm{Re}\,\lambda > 0. \tag{13}$$

If we continue this equation from positive to negative λ and consider the kernel of that integral as a generalised function (see also eqs. (56) - (61) of [13]), for $\lambda \to -n/2$ we obtain

$$\frac{\Gamma(\lambda+ik)\,\Gamma(\lambda-ik)}{2\pi\Gamma(2\lambda)} \quad\longrightarrow\quad \sum_{m=0}^{n} \binom{n}{m} \delta(k - i(n/2-m)). \tag{14}$$

In this manner (12) becomes in fact identical with the corresponding finite sum of (11).

It might be interesting to notice here that the investigations initiated by the references [4, 5] are based on Liouville's second form of the general solution [1]. This approach led [4] to a parametrisation of the Liouville theory in terms of a canonically related [19] but pseudo-scalar free field which is asymptotically neither an *in*- nor an *out*-field, whereas the work of [5, 10] mainly treats the singular elliptic monodromy for which we do not know whether there exists a parametrisation in terms of a real free field at all.

2 Vertex operators

The quantum Liouville theory will be defined by canonically quantising the free field (4)

$$[q,p] = i\hbar, \qquad [a_m, a_n] = \hbar\, m\, \delta_{m+n,0}, \tag{15}$$

and requiring that the vertex operators are primary and local. Such a procedure gives an anomaly-free quantum Liouville theory only if additional quantum deformations are taken into consideration [4] - [8]. A quantum realisation of (11), consistent with the operator Liouville equation and the canonical commutation relations, was so constructed in [6]. But the infinite sum is not a useful vertex operator. It is its finite form for negative integer $2\lambda = -n$ which presents a well-defined basis for the calculation of correlation functions.

Let us review the needed elements, and call $V_{2\lambda}(\tau,\sigma)$ the vertex operator of $e^{2\lambda\varphi(\tau,\sigma)}$. Taking, for simplicity, the same notations for the classical and the corresponding normal ordered quantum expressions, the vertex operator for $\lambda = -1/2$ can be written as

$$V_{-1}(\tau,\sigma) = e^{-\phi(\tau,\sigma)} + \mu_\alpha^2\, \frac{1}{2\sinh \pi P}\, e^{-\phi(\tau,\sigma)}\, S(\tau,\sigma)\, \frac{1}{2\sinh\pi P}. \tag{16}$$

Here μ_α^2 is the renormalised 'cosmological constant' and we introduced short notations

$$\mu_\alpha^2 = \mu^2\, \frac{\sin \pi\alpha}{\pi\alpha}, \qquad \alpha = \frac{\hbar\gamma^2}{2\pi}, \qquad P = \frac{\gamma p}{2\pi}, \tag{17}$$

and defined the conformal weight zero screening charge operator as

$$S(\tau,\sigma) = e^{\gamma q + P\tau} A_p(x) \bar{A}_p(\bar{x}) e^{\gamma q + P\tau}, \tag{18}$$

where

$$A_p(x) = \int_0^{2\pi} dy\, e^{2\phi(x+y) + P(y-\pi)} \tag{19}$$

is the integral (7) rewritten by using the periodicity and $\epsilon(z) = sign(z)$ for $z \in (-2\pi, 2\pi)$.

The useful factorised form of the operator V_{-n} can be constructed easily by induction

$$V_{-n-1}(x,\bar{x}) = \lim_{\epsilon \to 0} V_{-n}(x,\bar{x})\, V_{-1}(x+\epsilon, \bar{x}-\epsilon)\, \epsilon^{\alpha\, n}, \tag{20}$$

where the regularising factor $\epsilon^{\alpha\, n}$ just removes the short distance singularity, and it results

$$V_{-n}(\tau,\sigma) = \sum_{m=0}^{n} C_m^n\, \mu_\alpha^{2m} \prod_{l=1}^{m} \frac{1}{2\sinh \pi(P + i\, l\alpha)}\, e^{-n\phi(\tau,\sigma)}\, S^m(\tau,\sigma) \prod_{l=1}^{m} \frac{1}{2\sinh\pi(P - i\, l\alpha)}. \tag{21}$$

The shift of the momenta is a consequence of locality, and the same holds for the deformed binomial coefficients

$$C_m^n = \prod_{l=1}^{m} \frac{\sin \pi(n-l+1)\alpha}{\sin \pi l\alpha}, \tag{22}$$

whereas the hidden short distance contributions of (21) are due to the conformal properties.

Note that $[e^{-\phi(\tau,\sigma)}, S(\tau,\sigma)] = 0$ provides hermiticity of (16). The vertex operators for arbitrary λ will be described jointly with the correlation functions in the next section.

3 Correlation functions

Owning to conformal invariance we have to calculate 3-point correlation functions only. They are defined by matrix elements of vertex operators $\langle p;0|\, V_{2\lambda}(0,0)\,|p';0\rangle$ between the highest weight vacuum state $|p;0\rangle$ ($p>0$) which gets annihilated by the operators a_n with $n>0$. Using (21) the correlation function for $2\lambda=-n$ becomes ($P=\frac{\gamma p}{2\pi}$!)

$$\langle p;0|\, V_{-n}(0,0)\,|p';0\rangle = \sum_{m=0}^{n} C_m^n\, \mu_\alpha^{2m}\; J_m^n(P,\alpha)\; (I_m^n(P,\alpha))^2\; \delta(P-P'-i(n-2m)\alpha), \quad (23)$$

where $J_m^n(P,\alpha)$ summarises the p-dependent factors of the sinh-terms of (21)

$$J_m^n(P,\alpha) = \prod_{l=1}^{m} \frac{1}{4\,\sinh \pi(P+il\alpha)\;\sinh \pi(P-i(n-2m+l)\alpha)}, \quad (24)$$

and $I_m^n(P,\alpha)$ is the (anti-)chiral matrix element $\langle\, 0|\, e^{-n\phi}\, \prod_{l=1}^{m} A_{P-i(n-2l+1)\alpha}\, |0\,\rangle$ which is given by the integrals over the conformal short-distance deformations of (21)

$$I_m^n(P,\alpha) = \int_0^{2\pi} dy_1 \ldots \int_0^{2\pi} dy_m \prod_{l=1}^{m} e^{(P-\frac{i}{2}n\alpha+im\alpha)(y_l-\pi)} \left(2\sin\frac{y_l}{2}\right)^{n\alpha} \times \quad (25)$$

$$\prod_{1\le k<l\le m} \left(4\sin^2\frac{y_k-y_l}{2}\right)^{-\alpha} e^{i\pi\epsilon(y_k-y_l)}.$$

Fortunately these integrals can be expressed [10] by Dotsenko-Fateev integrals [20]

$$I_m^n(P,\alpha) = \prod_{l=1}^{m} \frac{\Gamma(1+(n-l+1)\alpha)}{\Gamma(1+l\alpha)}\; \frac{2\pi\;\Gamma(1+\alpha)}{\Gamma(1+l\alpha-iP)\Gamma(1+(n-2m+l)\alpha+iP)}. \quad (26)$$

If we replace the sin- (sinh-) functions of (22) (respectively (24)) by Γ-functions

$$\frac{\pi x}{\sin \pi x} = \Gamma(1+x)\,\Gamma(1-x), \quad (27)$$

we find for the correlation function the result

$$\langle p;0|\, V_{-n}(0,0)\,|p';0\rangle = \sum_{m=0}^{n} \binom{n}{m} \left(\mu^2 \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right)^m V_m^n(P,P')\,\delta(P-P'-i(n-2m)\alpha), \quad (28)$$

with

$$V_m^n(P,P') = \prod_{l=1}^{m} \frac{\Gamma(iP-l\alpha)\;\Gamma(-iP'-l\alpha)\;\Gamma(1+(n-l+1)\,\alpha)\Gamma(1-l\alpha)}{\Gamma(1-iP+l\alpha)\,\Gamma(1+iP'+l\alpha)\,\Gamma(1+(l-n-1)\,\alpha)\,\Gamma(1+l\alpha)}. \quad (29)$$

Our aim is to continue these functions from the negative value $2\lambda=-n$ to positive λ. We apply eq. (14) and obtain, with $2\alpha k = P-P'$ from the continued δ-function of (28),

$$\langle p;0|\, V_{2\lambda}(0,0)\,|p';0\rangle = \frac{\Gamma\,(\lambda+ik)\;\Gamma\,(\lambda-ik)}{4\pi\alpha\,\Gamma(2\lambda)} \left(\mu^2\;\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right)^{ik-\lambda} V_{ik-\lambda}^{-2\lambda}(P,P'), \quad (30)$$

where $V^{-2\lambda}_{ik-\lambda}(P, P')$ is the analytical continuation of (29). This continuation will be performed by means of the integral representation of the Γ-function [18]

$$\log \Gamma(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-t}}{1 - e^{-t}} + (x-1)e^{-t} \right], \tag{31}$$

and the following summation under that integral

$$f(x,\, \alpha |\, m) = \sum_{l=0}^{m-1} \log \Gamma(x + l\alpha) = \tag{32}$$

$$\int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt}\,(1 - e^{-m\alpha t})}{(1 - e^{-t})(1 - e^{-\alpha t})} - \frac{m e^{-t}}{1 - e^{-t}} + m(x-1)e^{-t} + \alpha \frac{m(m-1)}{2}\, e^{-t} \right].$$

The function $f(x,\, \alpha |\, m)$ has a natural analytical continuation with respect to α, m and x, and the useful property $f(x - m\alpha,\, \alpha |\, m) = f(x - \alpha,\, -\alpha |\, m)$. To simplify our result we rewrite a factor of (29)

$$\prod_{l=1}^{m} \frac{1}{\Gamma(1 + (l - n - 1)\,\alpha)} = \frac{1}{\alpha^m} \prod_{l=1}^{m} \frac{1}{l - n - 1} \prod_{l=1}^{m} \frac{1}{\Gamma((l - n - 1)\,\alpha)}, \tag{33}$$

and continue it separately

$$\frac{1}{\alpha^{ik-\lambda}} \frac{\Gamma(2\lambda)}{\Gamma(ik + \lambda)}\, e^{-f(2\lambda\alpha,\, \alpha |\, m)}. \tag{34}$$

After analytically continuing the remaining terms of (29), and obvious cancellations, we obtain finally the generic correlation function we are looking for as

$$\langle p; 0 |\, V_{2\lambda}(0,0)\, | p'; 0 \rangle = \frac{\Gamma(\lambda - ik)}{4\pi\alpha} \left(\frac{\mu^2}{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} \right)^{ik-\lambda} \prod_{j=1}^{4} e^{f(x_j,\, \alpha |\, m) - f(y_j,\, \alpha |\, m)}, \tag{35}$$

where

$$\begin{array}{llll} x_1 = iP - m\alpha, & x_2 = -iP' - m\alpha, & x_3 = 1 + (n - m + 1)\alpha, & x_4 = 1 - m\alpha; \\ y_1 = 1 + \alpha - iP, & y_2 = 1 + \alpha + iP', & y_3 = -n\alpha, & y_4 = 1 + \alpha; \end{array}$$

$$m = ik - \lambda, \qquad n = -2\lambda, \qquad k = \frac{P - P'}{2\alpha}, \qquad \alpha = \frac{\hbar\gamma^2}{2\pi}, \qquad P = \frac{\gamma p}{2\pi}. \tag{36}$$

It is worth mentioning here that the function (32) was used for the parametrisation of the 3-point correlation functions suggested in [14], and it is easy to show that with eqs. (35)- (36) we have re-derived that result (see eq. (14) of the second reference of [14] with obvious changes of the notation and overall renormalization).

However, there are some further remarks in order. Dorn and Otto [14] started their analytical continuation from a path-integral result for minimal models [15] which is just the one term $n = 2m$ of the operator calculated correlation function (28) proportional to

$\delta(P-P')$. This single term would be selected in our calculations if and only if screening charge conservation could be operative [9]. However, the Liouville theory is Möbius non-invariant and all the $(n+1)$ terms of (28) together characterise this theory for $2\lambda=-n$. Moreover, for odd n the correlation functions of [14] vanish and only the neglected n terms guarantee the necessary non-vanishing of the Liouville correlation functions in those points. Vice versa, by analytically continuing the correlation function of [14] as generalised function, in the manner explicitely described for the zero modes in ref. [13], one finds the correlation function for negative λ which for $\lambda=-n/2$ just reproduces (28).

We should, furthermore, mention that the functions (32) and Υ_b of [16] are related by

$$
\begin{aligned}
f(bu,\, b^2\,|s) - f(bv,\, b^2\,|s) &= \log\Upsilon_b(v) - \log\Upsilon_b(u) + sb(u-v)\log b,\\
f(1-sb^2,\, b^2\,|s) - f(1+b^2,\, b^2\,|s) &= \log\frac{\Upsilon_b(1/b)}{\Gamma(-s)\,\Upsilon_b(-sb)} + (s+1)(1-sb^2)\log b,
\end{aligned}
\tag{37}
$$

where $b^2=\alpha$, and $u+v=Q-sb$ with $Q=b+1/b$. With these equations we can derive from (35) the more heuristically motivated alternative, but to [14] equivalent, correlation functions of [16] as follows

$$
4\pi\alpha\langle p;0|\,V_{2\lambda}(0,0)\,|p';0\rangle = C(\,\frac{1}{2}\,(Q-iP/b)\,,\ b\lambda,\ \frac{1}{2}\,(Q+i\,P'/b)\,). \tag{38}
$$

By the same procedure of analytical continuation as used before, and by taking into consideration the results of this section, we obtain from (21) the vertex operator for positive λ as an integral representation

$$
\begin{aligned}
V_\lambda(\tau,\sigma) = \int_{-\infty}^{+\infty} dk\ \frac{\Gamma(\lambda+ik)\,\Gamma(\lambda-ik)}{2\pi\Gamma(2\lambda)}\ \left(\frac{\mu^2}{2\pi\alpha\,\Gamma(1+\alpha)\Gamma(1-\alpha)}\right)^{ik-\lambda} \times \\
Y_\alpha(\lambda,\,k)\,X_a(p,\,ik-\lambda)\,e^{2\lambda\phi(\tau,\sigma)}\,S^{ik-\lambda}(\tau,\sigma)\,X^*_a(p,\,ik-\lambda),
\end{aligned}
\tag{39}
$$

with

$$
Y_\alpha(\lambda,k) = \frac{e^{f(1+\alpha,\,\alpha|ik-\lambda)+f(1-(ik-\lambda)\alpha,\,\alpha|ik-\lambda)}}{e^{f(1+2\alpha\lambda,\,\alpha|ik-\lambda)+f(1-(\lambda+ik-1)\alpha,\,\alpha|ik-\lambda)}}, \tag{40}
$$

$$
X_a(p,\lambda) = e^{f(1+iP-(ik-\lambda)\alpha,\,\alpha|ik-\lambda)+f(1-iP+\alpha,\,\alpha|ik-\lambda)}\ \frac{\Gamma(1+ik-\lambda-iP/\alpha)}{\Gamma(1-iP/\alpha)}. \tag{41}
$$

But we should emphasize here that we do not have so far a recipe at hand to calculate the Liouville correlation functions for positive λ directly from this integral. It would require the knowledge of complex powers of the screening charge operator $S^{ik-\lambda}(\tau,\sigma)$, which incidentally would give the S-matrix in compact form too. But this problem is at present under investigation only, and that is the reason why we cannot compare our vertex operator (39) with the Ansatz of [11] for which corresponding screening charge operators are not given either, and as well no recipe how to treat that vertex for an operator calculation of correlation functions directly.

4 Conclusions

With a suitable free-field parametrisation ad hand, canonical quantisation proves to be a straightforward and reliable approach for a description of the quantum Liouville theory. We have calculated the correlation functions for generic vertex operators by using known algebraic quantum structures of the theory and their distributional properties. But the derived integral vertex operators could not be applied directly since the complex powers of screening charge operators are not yet constructed. The exact S-matrix is therefore not available too. Nevertheless, it is known that self-adjointness of the Liouville theory as well as the reflection amplitudes follow from the S-matrix [17], which can be derived at least level by level. We can so conclude that we have got, in principle, a complete understanding of quantum Liouville theory. The remaining problems to be solved are mostly of technical nature.

Since the Liouville theory and the $SL(2;R)/U(1)$ respectively $SL(2;R)/R_+$ black hole cosets can be derived from the same $SL(2;R)$ WZNW theory by Hamiltonian reduction [21], we expect also a joint quantum treatment of them. While doing so, the Liouville theory is an important ingredient of the other cosets, so that its quantum description is a prerequisite for the quantisation of the other cosets. This might be relevant for AdS_3 and string theory too, and different boundary conditions should be taken into consideration.

Moreover, we believe that the observed causal non-equal time structures of the cosets are important for field theory in general, and that these two-dimensional conformal field theories will remain outstanding examples of mathematical physics even in the next future.

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